

DECOMPOSITION NUMBERS FOR BRAUER ALGEBRAS OF TYPE $G(m, p, n)$ IN CHARACTERISTIC ZERO

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ABSTRACT. We introduce Brauer algebras associated to complex reflection groups of type $G(m, p, n)$, and study their representation theory via Clifford theory. In particular, we determine the decomposition numbers of these algebras in characteristic zero.

INTRODUCTION

The symmetric and general linear groups satisfy a double centraliser property over tensor space. This relationship is known as Schur–Weyl duality and allows one to pass information between the representation theories of these algebras. The Brauer algebra was defined to play the role of the symmetric group algebra in a Schur–Weyl duality with the orthogonal (or symplectic) group.

The original definition of the Brauer algebra has been generalised in many directions (see for example [BW89, HO01, CFW09, CLY, Tur89, Koi89]). In this paper we regard the classical Brauer algebra, $B_n(\delta)$ as an enlargement of the symmetric group algebra; in other words it corresponds to an enlargement of a complex reflection group of type $G(1, 1, n)$. By considering analogous enlargements of other complex reflection groups, we arrive at the Brauer algebras of type $G(m, p, n)$.

The type $G(m, 1, n)$ case was studied in [BCD13], where the decomposition numbers for these algebras are calculated by a reduction to the type $G(1, 1, n)$ case. In this paper we study the Brauer algebras of type $G(m, p, n)$. Using a combination of diagram algebra techniques, Clifford theory, and Brauer–Humphreys reciprocity, we calculate the decomposition numbers of these algebras.

We begin in Section 1 by defining the Brauer algebras, $B_{m,p,n}$, of type $G(m, p, n)$ and realise the algebra of type $G(m, 1, n)$ as a skew group algebra. This will allow us to apply the methods of Clifford theory. We then review the basic representation theory of complex reflection groups which is both required for and motivates the results that follow.

In Section 3 we begin to study the representation theory of the Brauer algebras of type $G(m, p, n)$. We deduce when the algebra is quasi-hereditary and give explicit constructions of the standard modules. We then apply Clifford

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theory to deduce restriction rules for standard, simple, and projective modules. We briefly consider restriction to the underlying group algebra using Littlewood–Richardson theory.

Using Clifford theory and the fact that Hom-spaces for $B_{m,1,n}$ have nice rotational symmetries, we are able to decompose Hom-spaces for $B_{m,p,n}$. Combining these results and Brauer–Humphreys’ reciprocity, we conclude by determining the decomposition numbers of $B_{m,p,n}$ in terms of those for the classical Brauer algebra (which have been given in terms of Kazhdan–Lusztig polynomials by [Mar]).

1. BRAUER ALGEBRAS OF TYPE $G(m, p, n)$

We fix k , an algebraically closed field. Let $m, p, n \in \mathbb{N}$ be such that $pd = m$ for some $d \in \mathbb{N}$. In this section we will define the Brauer algebras, $B_{m,p,n}$, of type $G(m, p, n)$. We shall show that the Brauer algebra of type $G(m, p, n)$ is a subalgebra of that of type $G(m, 1, n)$ introduced in [BCD13, Appendix] (where it was called the unoriented cyclotomic Brauer algebra).

1.1. Definitions. Given $n \in \mathbb{N}$ and $\delta = (\delta_0, \delta_p, \delta_{2p}, \dots, \delta_{(d-1)p}) \in k^d$, the *Brauer algebra of type $G(m, p, n)$* , denoted by $B_{m,p,n}$, is a finite dimensional associative k -algebra generated by certain Brauer diagrams. A *diagram* consists of a *frame* with n distinguished points on the northern and southern boundaries, which we call *nodes*. Each node is joined to precisely one other by a strand; strands connecting the northern and southern edge will be called *through-strands* and the remainder (*northern or southern*) *arcs*. There may also be closed loops inside the frame, those diagrams without closed loops are called *reduced* diagrams.

Each strand is labelled by an element of the cyclic group $\mathbb{Z}/m\mathbb{Z}$; we require the additional restriction that the total sum over the labels is a multiple of p . When drawing diagrams we will adopt the convention that unlabelled arcs have label 0. Two diagrams are equivalent if the strands connect the same pairs of nodes and have the same labels. As a vector space, $B_{m,p,n}$ is the k -span of the reduced diagrams. Figure 1 gives an example of two such elements in $B(6, 3, 6)$.



FIGURE 1. Two elements in $B_{6,3,6}(\delta)$

Given $x, y \in B_{m,p,n}$, we define the product $x \cdot y$ to be the diagram obtained by concatenation of x above y , where we identify the southern nodes of x with the northern nodes of y and then ignore the section of the frame common to both diagrams.

The label of each strand, s , in the concatenated diagram, is then the sum of the labels of the strands it is composed from. The product of two diagrams may contain a closed loop: if this loop is labelled by $ip \in \mathbb{Z}/m\mathbb{Z}$ then the diagram

is set equal to δ_{ip} times the same diagram with the loop removed; if the label is not divisible by p , we set the product to be zero.

Example 1.1.1. The product $x \cdot y$ of the elements in Figure 1 is given in Figure 2. The product $y^2 = 0$ as it results in the removal of a closed loop labelled by 2 (when reduced mod 6), which is not divisible by 3.

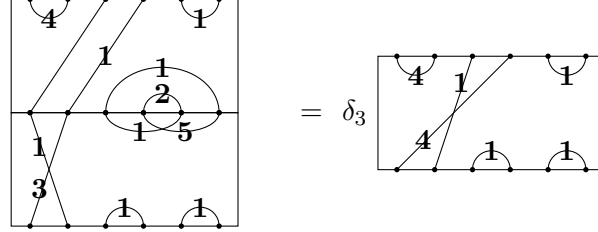


FIGURE 2. The product $x \cdot y$

We will need to speak of certain elements of the algebra with great frequency. The elements $s_{i,j}$, t_i^k , $s_{i,j}^*$, and $e_{i,j}$ (for $i, j \leq n$) are indicated in Figure 3, where the nodes are numbered in increasing order from left to right by 1 up to n on the northern edge, and $\bar{1}$ up to \bar{n} on the southern edge.

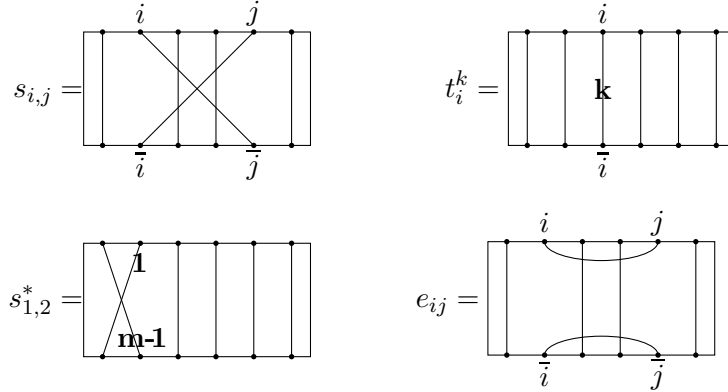


FIGURE 3. The elements $s_{i,j}$, t_i^k , and $s_{i,j}^*$ and $e_{i,j}$

Remark 1.1.2. The $p = 1$ case was first studied in the Appendix to [BCD13]. There it is christened the *un-oriented cyclotomic Brauer algebra*; this algebra is not the (oriented) cyclotomic Brauer algebra studied in [AMR06] and elsewhere. Both the oriented and un-oriented cyclotomic Brauer algebras are specialisations of the BMW algebra. However, it is only the un-oriented algebra which has a family of subalgebras which can be studied by analogy with the complex reflection groups of type $G(m, p, n)$.

1.2. Clifford theory I. Consider the algebra $B_{m,1,n}$, as defined above. Let $p|m$ and specialise the parameter $\delta \in k^m$ so that δ_i is zero for any index i that is not congruent to zero modulo p , i.e. take

$$\delta = (\delta_0, \dots, 0, \delta_p, 0, \dots, 0, \delta_{2p}, 0, \dots, 0, \delta_{p(d-1)}, 0, \dots, 0) \in k^m.$$

Take the subspace of $B_{m,1,n}$ (with parameter as above) spanned by all diagrams whose labels sum to a multiple of p . Multiplication is inherited from that in $B_{m,1,n}$; our choice of parameter ensures that any closed loops removed are labelled by a multiple of p (otherwise the product is zero) and therefore the diagram obtained by their removal still lies in the same subspace. Therefore this subspace is in fact a subalgebra, and is clearly isomorphic to $B_{m,p,n}$. Throughout this paper, we shall only consider $B_{m,1,n}$ for the parameter as above.

Let $\mathbb{Z}/p\mathbb{Z}$ act via the k -algebra automorphism of $B_{m,1,n}$ given by conjugation by t_1^d . This maps $B_{m,p,n}$ onto $B_{m,p,n}$. We have the following theorem.

Theorem 1.2.1. *The algebra $B_{m,1,n}$ (with parameter δ as above) is the skew group algebra*

$$B_{m,1,n} = B_{m,p,n} \rtimes \mathbb{Z}/p\mathbb{Z} = \left\{ \sum_{z \in \mathbb{Z}/p\mathbb{Z}} d_z z : d \in B_{m,p,n} \right\}$$

with linear multiplication given by the concatenation action: $zd = (zdz^{-1})z$.

Proof. This is similar to the the group algebra case. The natural diagram basis of $B_{m,1,n}$ can be partitioned into p distinct sets, $\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_{p-1}$, (of equal cardinality) each consisting of the diagrams whose sum over the labels is congruent to $0, 1, \dots, p-1$ modulo p respectively. The algebra $B_{m,p,n}$ has basis given by \mathcal{B}_0 , as seen above.

Left and right multiplication by t_1 both define bijections from \mathcal{B}_i to \mathcal{B}_{i+1} . Using this to rewrite diagrams as $t_1^i d'$ or as $d'' t_1^j$ for some $d', d'' \in \mathcal{B}_0$, one can check that the multiplication on $B_{m,p,n} \rtimes \mathbb{Z}/p\mathbb{Z}$ is equivalent to the multiplication on $B_{m,1,n}$ (with δ as above). \square

This result means that we will later be able to apply methods from Clifford theory (see [RR85]).

1.3. Generators for subalgebras. Just as for the Brauer algebra, it follows from the definitions that $B_{m,1,n}$ is generated by $s_{i,i+1}$, t_1 and $e_{1,2}$. The group algebra of type $G(m, p, n)$ can be identified with the subalgebra of $B_{m,1,n}$ generated by $s_{1,2}^*$, $s_{i,i+1}$ and t_1^p ; for $n > 2$ the subalgebra $B_{m,p,n}$ of $B_{m,1,n}$ is generated by $s_{1,2}^*$, $s_{i,i+1}$, t_1^p and $e_{1,2}$ for $1 \leq i \leq n-1$.

1.4. Cyclotomic parameters. We have defined the algebra $B_{m,p,n}$ in terms of $\delta = (\delta_0, \delta_p, \dots, \delta_{(d-1)p}) \in k^d$. It is shown in [BCD13] that the following cyclotomic functions of these parameters govern the representation theory of the algebra $B_{m,1,n}$ under the assumption that m is invertible in k .

Definition 1.4.1. For each $0 \leq r \leq d-1$ we define the r th cyclotomic parameter to be

$$\bar{\delta}_r = \frac{1}{m} \sum_{i=0}^{d-1} \xi^{ipr} \delta_{ip}.$$

where $\xi \in k^\times$ is a primitive m th root of unity

2. REFLECTION GROUPS OF TYPE $G(m, p, n)$

We have already assumed that k is an algebraically closed field. Henceforth we shall also assume that k is of characteristic zero and we shall fix $\xi \in k^\times$, a primitive m th root of unity. The group algebra of the complex reflection group, $G(m, 1, n)$, is the *skew group algebra*

$$G(m, 1, n) = G(m, p, n) \rtimes \mathbb{Z}/p\mathbb{Z},$$

this comes from taking the semidirect product of the two groups. We shall study $G(m, p, n)$ via Clifford theory. The results in this section can be found in [MM10, Section 2.3].

2.1. Type $G(m, 1, n)$ combinatorics. A partition is a finite weakly-decreasing sequence of non-negative integers. An m -partition of n is an m -tuple of partitions $\lambda = (\lambda^0, \dots, \lambda^{m-1})$ such that $\sum_{i=0}^{m-1} |\lambda^i| = n$ (where $|\lambda^i|$ denotes the sum of the parts of the partition λ^i). We let $\Lambda(m, 1, n)$ denote the set of all m -partitions of $n - 2l$ for $l \leq n/2$; we let $\Lambda_0(m, 1, n)$ denote the subset where $l = 0$.

Let λ be an m -partition of n . A λ -tableau is a bijection $\mathbf{t} : \lambda \rightarrow \{1, 2, \dots, n\}$, which we consider as an m -tuple $\mathbf{t} = (\mathbf{t}^0, \dots, \mathbf{t}^{m-1})$ of labelled tableaux where \mathbf{t}^s is a λ^s -tableau for each s ; the tableaux \mathbf{t}^s are the components of \mathbf{t} . We say a tableau, \mathbf{t} , is *standard* if the entries in the component tableaux are increasing along the rows and columns. We let \mathcal{T}_λ denote the set of standard λ -tableaux.

For \mathbf{t} a tableau, we set $\mathbf{t}(i) = s$ if the integer i appears in \mathbf{t}^s . Let $1 \leq i < j \leq n$, we define the axial distance, $a(i, j)$, as follows: if $\mathbf{t}(i) \neq \mathbf{t}(j)$ then $a(i, j) = \infty$ (so that $1/a(i, j) = 0$); if $\mathbf{t}(i) = \mathbf{t}(j)$ and i occurs in row i_0 and column i_1 and j occurs in row j_0 and column j_1 , then $a(i, j) = (i_0 - i_1) - (j_0 - j_1)$.

If \mathbf{t} is a λ -tableau and $w \in \Sigma_n$ let $w\mathbf{t}$ be the tableau obtained from \mathbf{t} by replacing each entry in \mathbf{t} by its image under w . Let $\mathbf{t} \in \mathcal{T}_\lambda$, we set $\mathbf{t}_{i \leftrightarrow i+1}$ equal to $s_{i, i+1}\mathbf{t}$ if this is still a standard λ -tableau, and 0 otherwise.

Proposition 2.1.1. *The algebra $kG(m, 1, n)$ has simple modules indexed by the poset $\Lambda_0(m, 1, n)$. For a given m -partition λ of n , the simple module $\overline{\mathbf{S}(\lambda)}$ has a basis given by the set of standard λ -tableaux. With respect to this basis the generators act as follows*

$$\overline{\rho_\lambda}(t_1)\mathbf{t} = \xi^{\mathbf{t}(1)}\mathbf{t}, \quad \overline{\rho_\lambda}(s_{i, i+1})\mathbf{t} = \frac{1}{a(i, i+1)}\mathbf{t} + \left(1 + \frac{1}{a(i, i+1)}\right) \mathbf{t}_{i \leftrightarrow i+1}$$

2.2. Type $G(m, p, n)$ combinatorics. Let $pd = m$ and let σ be a distinguished generator of $\mathbb{Z}/p\mathbb{Z}$. There is a natural action of the cyclic group $\mathbb{Z}/p\mathbb{Z}$ on the poset $\Lambda(m, 1, n)$ given by permutation of the indices. This extends to an action on tableaux by setting

$$\sigma : (\mathbf{t}^0, \mathbf{t}^1, \dots, \mathbf{t}^{m-1}) \mapsto (\mathbf{t}^{m-d}, \mathbf{t}^{1-d}, \dots, \mathbf{t}^{m-1-d}),$$

we denote $\sigma(\mathbf{t}) = \mathbf{t}^\sigma$.

For $\lambda \in \Lambda(m, 1, n)$ let $\text{Stab}(\lambda)$ denote the stabiliser of λ under the permutation action. We have $\text{Stab}_{\mathbb{Z}/p\mathbb{Z}}(\lambda) = \langle \sigma^t \rangle$ and $0 \leq r < p/t$. We let $\Lambda(m, p, n)$ denote the set of pairs consisting of a representative of a $\mathbb{Z}/p\mathbb{Z}$ -orbit on $\Lambda(m, 1, n)$ and an integer $0 \leq r < p/t$. We let $\Lambda_0(m, p, n)$ denote the subset where $l = 0$.

Example 2.2.1. We have that $\Lambda(2, 1, 2) = \{(\emptyset, \emptyset), (\emptyset, 2), (\emptyset, 1^2), (2, \emptyset), (1^2, \emptyset)\}$. There is a unique element, (\emptyset, \emptyset) , with non-trivial stabiliser $\langle \sigma^1 \rangle = \mathbb{Z}/2\mathbb{Z}$. Therefore $\Lambda(2, 2, n)$ has four elements and (picking a set of orbit representatives) is equal to the set $\{(\emptyset, \emptyset)^0, (\emptyset, \emptyset)^1, (2, \emptyset), (1^2, \emptyset)\}$.

2.3. Simple modules for $G(m, p, n)$. We now give the construction, via Clifford theory, of the simple modules for $G(m, p, n)$. We do not go into much detail here, and instead refer to [MM10].

Simple modules for $G(m, p, n)$ are labelled by a representation of $\text{Stab}(\lambda) \leq \mathbb{Z}/p\mathbb{Z}$ (given by an integer $0 \leq r < p/t$) and a representation of $G(m, 1, n)$ (given by an m -partition). Recall that $\text{Stab}(\lambda) = \langle \sigma^t \rangle \leq \mathbb{Z}/p\mathbb{Z}$. By Clifford theory, we have that $\overline{\mathbf{S}(\lambda)} \downarrow = \oplus_{0 \leq r < p/t} \mathbf{S}(\lambda^r)$, where

$$\mathbf{S}(\lambda^r) = \ker(\sigma^t - \xi^{dtr}) \overline{\mathbf{S}(\lambda)}$$

for $0 \leq r < p/t$. We let

$$p_r = \frac{t}{p} \sum_{0 \leq i < p/t} \xi^{-idtr} \sigma^{it}$$

denote the projection onto this subspace.

Take as representatives of the $\langle \sigma^t \rangle$ -orbits the $\mathbf{t} \in \mathcal{T}_\lambda^0$ where \mathcal{T}_λ^0 is the set of standard λ -tableaux with $\mathbf{t}(1) < td$. Take the subspace spanned by tableaux in \mathcal{T}_λ^0 and apply the projection p_r , this provides a basis of $\mathbf{S}(\lambda^r)$ (in the case that $r = 0$ this is the average of the $\langle \sigma^t \rangle$ -orbit). Setting $\mathbf{t}^r = p_r(\mathbf{t})$, we then get formulae for the action of the generators of $G(m, p, n)$ on $\mathbf{S}(\lambda^r)$ as follows: $\overline{\rho_\lambda}(t)p_r = p_{r+1}\overline{\rho_\lambda}(t)$, and so

$$\rho_{\lambda,r}(t^p)\mathbf{t}^r = \xi^{prt(1)}\mathbf{t}^r, \quad \rho_{\lambda,r}(s_1^*)\mathbf{t}^r = \xi^{r(\mathbf{t}(1)-\mathbf{t}(2))}\rho_{\lambda,r}(s_1)\mathbf{t}^r,$$

$$\rho_{\lambda,r}(s_i)\mathbf{t}^r = \frac{1}{a(i, i+1)}\mathbf{t}^r + \left(1 + \frac{1}{a(i, i+1)}\right)\mathbf{t}_{i \leftrightarrow i+1}^r.$$

3. BRAUER ALGEBRAS OF TYPE $G(m, p, n)$

Unless otherwise stated, let k denote an algebraically closed field of characteristic zero. Fix ξ to be a primitive m th root of unity. In this section we study $B_{m,p,n}$ via Clifford theory. By verifying the first two conditions of a tower of recollement, we deduce conditions under which the algebra is quasi-hereditary; we leave it as an exercise for the reader to check that the algebra is a tower of

recollement (in the sense of [CMPX06]) by checking that it obeys the remaining conditions.

3.1. Highest weight theory. Let $n \geq 2$. Suppose first that $\delta \neq \mathbf{0} \in \mathbf{k}^d$ and fix a $\delta_{ip} \neq 0$ for some $0 \leq i < d$. We then define the idempotent $e_{n-2} = \frac{1}{\delta_{ip}} t_{n-1}^{ip} e_{n-1, n}$ as illustrated in Figure 4. Note that it is a scalar multiple of a diagram with $n - 2$ through-strands. If $\delta = 0$ and $n \geq 3$ then we define e_{n-2} to be the idempotent $e_{n-1, n} e_{n-2, n-1}$, as illustrated in Figure 4.



FIGURE 4. The idempotent e_{n-2} (for $n = 6$) in the cases that $\delta \neq 0$, $\delta = 0$ respectively.

A tower of recollement was defined in [CMPX06] to be a family of algebras (with idempotents) satisfying six conditions (A1–6). It is easy to see that

$$(3.1.1) \quad e_{n-2} B_{m, p, n} e_{n-2} \cong B_{m, p, n-2}$$

and that

$$(3.1.2) \quad B_{m, p, n} / B_{m, p, n} e_{n-2} B_{m, p, n} \cong kG(m, p, n).$$

For the latter isomorphism, note that the lefthand-side has a basis consisting of the diagrams with no arcs. Therefore we have the following

Theorem 3.1.1. *Let k be a field of characteristic $\text{cha}(k) \geq 0$. Let $m, n \in \mathbb{N}$, and $\delta \in k^m$. If n is even suppose $\delta \neq \mathbf{0} \in k^m$. The algebra $B_{m, p, n}(\delta)$ is quasi-hereditary if and only if $\text{cha}(k) > n$ and $\text{cha}(k) \nmid m$, or $\text{cha}(k) = 0$.*

We leave it to the reader to verify the remaining tower conditions using classical tower arguments (see [CDDM08], [CDM09]) and Clifford theory.

3.2. The standard modules of $B_{m, 1, n}$. Recall our assumption on the parameter $\delta \in k^m$ from Section 1.2. By [BCD13, Theorem 3.1.2], the algebra $B_{m, 1, n}$ is an iterated inflation of the group algebras $G(m, 1, n - 2l)$ along vector spaces V_l spanned by all possible (m, n, l) -tangles. An (m, n, l) -tangle has l arcs denoted by (i_p, j_p) (for $p = 1, \dots, l$) where i_p (resp. j_p) is the left (resp. right) vertex of the arc, and $n - 2l$ free lines. Each arc has a label given by an element $r \in \mathbb{Z}/m\mathbb{Z}$. For example a $(5, 7, 2)$ -tangle is depicted in Figure 5.

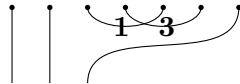


FIGURE 5. A $(5, 7, 2)$ -dangle

We therefore have the following theorem:

Theorem 3.2.1. *The algebra $B_{m,1,n}$ has standard modules indexed by $\Lambda(m, 1, n)$. For a given m -partition, λ , of $n - 2l$, we have the standard module*

$$\overline{\Delta(\lambda)} \cong V_l \otimes_k \overline{\mathbf{S}(\lambda)}.$$

The action of a diagram $X \in B_{m,1,n}$ on $v \otimes x \in \overline{\Delta(\lambda)}$ is given as follows. Apply the diagram X to the (m, n, l) -tangle v . If we obtain more than l arcs, or a closed loop labelled by an integer not divisible by p , this element is sent to zero. Otherwise, we obtain another (m, n, l) -tangle Xv and a signed permutation $\sigma \in G(m, 1, n - 2l)$ on the $n - 2l$ free vertices of Xv , we then define $X(v \otimes x) = (Xv) \otimes \sigma x$.

3.3. Standard modules for $B_{m,p,n}$. By (3.1.1) and (3.1.2), we have that the standard modules for $B_{m,p,n}$ are of the form

$$\Delta_n(\lambda^r) = (B_{m,p,n} / (B_{m,p,n} e_{n-2l-2} B_{m,p,n})) e_{n-2l} \otimes_{B_{m,p,n-2l}} \mathbf{S}(\lambda^r).$$

This module is spanned by the elements $d \otimes_{B_{m,p,n-2l}} \mathbf{t}^r$ where $\mathbf{t}^r \in \mathbf{S}(\lambda^r)$ and $d \in B_{m,p,n}$ with precisely $(n - 2l)$ through-lines. By taking elements of $B_{m,p,n-2l}$ across the tensor product we can just consider diagrams d with (a) no crossing through-lines (b) only the leftmost through-line has a non-zero label, (c) this label, q , is strictly less than p (as any diagram $d' \in B_{m,p,n}$ can be written as a product $d' = d\sigma$ for $\sigma \in G(m, p, n)$ and d of the required form). Of course, these diagrams must still be elements of $B_{m,p,n}$ and so the northern arcs of the diagram must have labels totalling $p - q$ modulo p . Figure 6 contains an example for type $G(6, 3, 7)$.

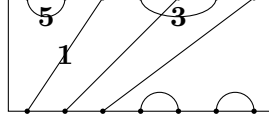


FIGURE 6. A diagram of type $G(6, 3, 7)$ satisfying condition (a), (b), and (c), above.

One can then pass the decoration on the left-most strand through the tensor product by noting that $t_1 \mathbf{t}^r = \mathbf{t}^{r+1}$ and that $t_1^{p/t} \mathbf{t}^r = \mathbf{t}^r$, by construction. Define $V_l(q, p/t) \subset V_l$ to be the subspace of dangles whose label sum is congruent to $-q$ modulo p/t .

Theorem 3.3.1. *The algebra $B_{m,p,n}$ has standard modules labelled by $\Lambda(m, p, n)$. For $\lambda^r \in \Lambda(m, p, n)$, we have that*

$$\Delta_n(\lambda^r) \cong \{v \otimes x : v \in V_l(q, p/t), x \in \mathbf{S}(\lambda^{q+r}), 0 \leq q < p/t\}$$

Example 3.3.2. The modules $\Delta((1, 0, 1, 0)^0)$ and $\Delta((1, 0, 1, 0)^1)$ for $B_{4,4,4}$ are both 24-dimensional. Let \mathbf{t} denote the unique element of $\mathcal{T}_{(1,0,1,0)}^0$ and let v be the dangle with a single undecorated arc $(1_p, 2_p)$. Some typical elements of $\Delta((1, 0, 1, 0)^0)$ are

$$v \otimes \mathbf{t}^0, \quad t_1^2 v \otimes \mathbf{t}^0, \quad t_1 v \otimes \mathbf{t}^1, \quad t_1^3 v \otimes \mathbf{t}^1,$$

and some typical elements of $\Delta((1, 0, 1, 0)^1)$ are

$$v \otimes \mathbf{t}^1, \quad t_1^2 v \otimes \mathbf{t}^1, \quad t_1 v \otimes \mathbf{t}^0, \quad t_1^3 v \otimes \mathbf{t}^0.$$

In fact, the bases of both modules can be obtained by applying undecorated elements of $G(4, 4, 2)$ to the elements above (i.e. by permuting the nodes of the dangles).

3.4. Clifford theory II. We will use Clifford theory techniques to give the decomposition of the restriction of a standard, simple, or projective module from $B_{m,1,n}$ to $B_{m,p,n}$.

3.4.1. Standard modules. Let (i_p, j_p) and (i_q, j_q) be two arcs in v with annotations l and k , respectively. We let $\epsilon_{l,ip}$ denote the Kronecker delta which is 1 or 0 if $l = ip$ for some $0 \leq i < d$, or not, respectively. We write $i \notin v$ if i labels a free line in v . Finally, note that there are n nodes on the top of a dangle and $n - 2l$ on the bottom of a dangle. If the i th node on the top of the diagram is a free node, we let \underline{i} denote the corresponding node on the bottom of the dangle.

From Theorem 3.2.1, we deduce that the action of the generators of $B_{m,1,n}$ (under our assumption on the parameter $\delta \in k^m$ from Section 1.2) on the standard module $\overline{\Delta(\lambda)}$ is as follows:

$$\begin{aligned} \pi_\lambda(t_1)(v \otimes \mathbf{t}) &= \begin{cases} \xi^{t(1)}(v \otimes \mathbf{t}) & \text{if } 1 \notin v \\ (t_1 v) \otimes \mathbf{t} & \text{if } 1 = i_p \text{ for some } p \end{cases} \\ \pi_\lambda(s_{i,i+1})(v \otimes \mathbf{t}) &= \begin{cases} \frac{1}{a(\underline{i}, \underline{i}+1)}(v \otimes \mathbf{t}) + \left(1 + \frac{1}{a(\underline{i}, \underline{i}+1)}\right)(v \otimes \mathbf{t}_{\underline{i} \leftrightarrow \underline{i}+1}) & \text{if } i, i+1 \notin v \\ (s_{i,i+1} v) \otimes \mathbf{t} & \text{otherwise} \end{cases} \\ \pi_\lambda(e_{1,2})(v \otimes \mathbf{t}) &= \begin{cases} 0 & \text{if } 1, 2 \notin v \\ \epsilon_{l,ip} \delta_{ip}((t_1^{-l} v) \otimes \mathbf{t}) & \text{if } 1 = i_p, 2 = j_p \\ \xi^{lt(1)} \pi_\lambda(s_{1,j_p})((t_1^{-l} v) \otimes \mathbf{t}) & \text{if } 1 \notin v, 2 = i_p \\ \pi_\lambda(s_{1,j_p})((t_1^l t_2^{-l} v) \otimes \mathbf{t}) & \text{if } 1 = i_q, 2 = i_p \end{cases} \end{aligned}$$

The case of $\pi_\lambda(e_{1,2})$ is symmetric in the coordinates 1, 2 and so we have omitted the details.

We recall that $\{t_1^p, s_1^*, e_{1,2}, s_{i,i+1} : 0 \leq q < r, 1 \leq i \leq n-1\}$ generate $B_{m,p,n}$ for $n > 2$, and that the quotient $B_{m,1,n}/B_{m,p,n}$ is cyclic, generated by t . Let χ be the generator of the group of linear characters of the quotient which maps t to ξ^d . From the formulae for the action of $t_1, s_{1,2}^*, e_{1,2}$, and the $s_{i,i+1}$ for $1 \leq i \leq n-1$, we see that the map

$$\sigma(v \otimes \mathbf{t}) = \xi^{dq} v \otimes \sigma(\mathbf{t}),$$

where q is the total label on v , induces an isomorphism $\chi \otimes \pi_\lambda = \pi_{\sigma(\lambda)}$.

It is easy to check that σ^t commutes with the action of $\rho_\lambda(s_{i,i+1})$, $\rho_\lambda(s_1^*)$ and $\rho_\lambda(e_{1,2})$, and that $\sigma^t \circ \rho_\lambda(t_1) = \xi^{dt} \rho_\lambda(t_1) \circ \sigma^t$. It follows that σ^t commutes with the action of $B_{m,p,n}$ and that

$$\overline{\Delta_n(\lambda)} \downarrow_{B_{m,p,n}}^{B_{m,1,n}} \cong \oplus_{0 \leq r < p/t} \ker(\sigma^t - \xi^{dtr}) \overline{\Delta_n(\lambda)},$$

(although these direct summands need not be indecomposable). For a given $0 \leq r < p/t$, we have that the projection onto $\ker(\sigma^t - \xi^{dtr})$ is given by:

$$p_r = \frac{t}{p} \sum_{0 \leq i < p/t} \xi^{-idtr} \sigma^{it}.$$

Theorem 3.4.1. *The restriction of a standard module, $\overline{\Delta(\lambda)}$, for $B_{m,1,n}$ is a direct sum of p/t standard modules for $B_{m,p,n}$. For $\lambda \in \Lambda(m, 1, n)$, we have that*

$$\Delta_n(\lambda^r) \cong \ker((\sigma^t - \xi^{dtr})\overline{\Delta_n(\lambda)}).$$

Proof. It suffices to show that $p_r \overline{\Delta(\lambda)}$ is the standard module constructed in the previous section. For a given $x \in V_l \otimes \overline{\mathbf{S}(\lambda)}$, we have that

$$\sigma^t(v \otimes \mathbf{t}) = \xi^{dqt} v \otimes \sigma^t(\mathbf{t})$$

where q is the label total on v , and therefore

$$p_r(v \otimes \mathbf{t}) = v \otimes \left(\frac{t}{p} \sum_{0 \leq i < p/t} \xi^{-idtr+dqt} \sigma^{it} \mathbf{t} \right) = v \otimes \mathbf{t}^{q+r}.$$

These elements form the basis of $\Delta(\lambda^r)$ given in Theorem 3.3.1, and the result follows. \square

3.4.2. Simple and projective modules. By Clifford theory [RR85, Theorems 1.1 and 1.3], we have that the simple $B_{m,1,n}$ -module, $\overline{L(\lambda)}$, restricts to a direct sum of simple $B_{m,p,n}$ -modules. As each simple $B_{m,1,n}$ -module appears as the head of the unique standard module with the same label, we have that

$$\overline{L(\lambda)} \downarrow_{B_{m,p,n}}^{B_{m,1,n}} \cong \oplus_{0 \leq r < p/t} L(\lambda^r),$$

by Frobenius reciprocity and Theorem 3.4.1. As $L(\lambda)$ is a quotient of $\overline{\Delta(\lambda)}$, we can use the action of σ^t on the quotient to characterise $L(\lambda^r)$ as $\ker(\sigma^t - \xi^{dtr})\overline{L(\lambda)}$.

The algebra, $B_{m,1,n}$, is free as a $B_{m,p,n}$ -module. Therefore the restriction of a projective module, is projective. By Frobenius reciprocity,

$$\overline{P(\lambda)} \downarrow_{B_{m,p,n}}^{B_{m,1,n}} \cong \oplus_{0 \leq r < p/t} P(\lambda^r).$$

For $\lambda \in \Lambda(m, 1, n)$, the projective module $\overline{P(\lambda)}$ appears as quotient (in fact, a direct summand) of

$$B(\lambda) = B_{m,1,n} e_{n-2l} \otimes_{B_{m,1,n-2l}} \overline{\mathbf{S}(\lambda)}.$$

We can therefore construct the projective modules as the eigenspaces of the automorphism σ^t (by first extending the σ^t -action to the module $B(\lambda)$ in the obvious way).

3.4.3. Restriction to the group algebra. We now calculate the structure of the projective, standard, and simple modules for $B_{m,p,n}$ upon restriction to $kG(m, p, n)$.

Proposition 3.4.2. *Let $\lambda^r, \mu^q \in \Lambda(m, p, n)$, with $\text{Stab}_{\mathbb{Z}/p\mathbb{Z}}(\lambda) = \langle \sigma^t \rangle$ and $\text{Stab}_{\mathbb{Z}/p\mathbb{Z}}(\mu) = \langle \sigma^u \rangle$. We have for a simple, projective, or standard $B_{m,p,n}$ -module $M(\lambda^r)$, that*

$$[M(\lambda^r) \downarrow_{kG(m,p,n)} : \mathbf{S}(\mu^q)] = \begin{cases} \sum_{\rho \in T} [\overline{M(\lambda)} : \overline{\mathbf{S}(\mu^\rho)}] & \text{if } r = q \text{ modulo } \text{hcf}(\frac{p}{t}, \frac{p}{u}) \\ 0 & \text{otherwise} \end{cases}$$

where T is a set of cosets for $\langle \sigma^{\text{hcf}(t,u)} \rangle \leq \mathbb{Z}/p\mathbb{Z}$.

Proof. The multiplicities $[\overline{M(\lambda)} : \overline{\mathbf{S}(\mu)}]$ are calculated in terms of Littlewood–Richardson coefficients in [BCD13, Appendix]. From this result, it is immediate that

$$[\overline{M(\lambda)} \downarrow_{G(m,1,n)} : \overline{\mathbf{S}(\mu)}] = [\overline{M(\lambda^\sigma)} \downarrow_{G(m,1,n)} : \overline{\mathbf{S}(\mu^\sigma)}].$$

We have that $\langle \sigma^t \rangle$ fixes $\overline{M(\lambda)}$ and $\langle \sigma^u \rangle$ fixes $\overline{\mathbf{S}(\mu)}$. Therefore

$$[\overline{M(\lambda)} \downarrow_{G(m,1,n)} : \mathbf{S}(\mu)] = [\overline{M(\lambda)} \downarrow_{G(m,1,n)} : \overline{\mathbf{S}(\mu^\tau)}]$$

for $\tau \in \langle \sigma^{\text{hcf}(t,u)} \rangle$. Therefore, we want to calculate the (well-defined) multiplicities

$$[\overline{M(\lambda)} \downarrow_{kG(m,p,n)} : (\oplus_{\tau \in \langle \sigma^{\text{hcf}(t,u)} \rangle} \overline{\mathbf{S}(\mu^\tau)}) \downarrow_{kG(m,p,n)}].$$

First, note that we can factorise the map $(\sigma^t - \xi^{dtr})$ as the product

$$(\sigma^t - \xi^{dtr}) = \prod_{0 \leq i < t} (\sigma - \xi^{d(r+ip/t)}).$$

Now, consider the kernel of the map $(\sigma^t - \xi^{dtr})$ applied to the direct sum. We have that

$$\begin{aligned} \ker(\sigma^t - \xi^{dtr})(\oplus_{\tau \in \langle \sigma^{\text{hcf}(t,u)} \rangle} \mathbf{S}(\mu^\tau)) &= \bigoplus_{0 \leq i < t} \ker(\sigma - \xi^{d(r+ip/t)})(\oplus_{\tau \in \langle \sigma^{\text{hcf}(t,u)} \rangle} \mathbf{S}(\mu^\tau)) \\ &= \bigoplus_{\substack{r=q \text{ mod} \\ \text{hcf}(p/t, p/u)}} \mathbf{S}(\mu^q). \end{aligned}$$

Summing over a set of coset representatives of $(\mathbb{Z}/p\mathbb{Z})/\langle \sigma^{\text{hcf}(t,u)} \rangle$ we obtain the desired result. \square

4. HOMOMORPHISMS BETWEEN STANDARD AND PROJECTIVE MODULES

Let $\lambda, \mu \in \Lambda(m, 1, n)$ with $\text{Stab}_{\mathbb{Z}/p\mathbb{Z}}(\lambda) = \langle \sigma^t \rangle$ and $\text{Stab}_{\mathbb{Z}/p\mathbb{Z}}(\mu) = \langle \sigma^u \rangle$. Let $0 \leq r < p/t$ and $0 \leq q < p/u$. We let $\lambda^r, \mu^q \in \Lambda(m, p, n)$ denote the elements corresponding to the r th and q th orbits.

Lemma 4.0.3. *Consider the algebra $B_{m,1,n}$ with parameter $\delta \in k^m$ as in Section 1.2. Let $\lambda, \mu \in \Lambda(m, 1, n)$. Let $\overline{M(\lambda)}$ be a standard or projective module labelled by λ . Let $\overline{N(\mu)}$ be a simple, standard, or projective module labelled by μ . We have that*

$$\text{Hom}_{B_{m,1,n}}(\overline{M(\lambda)}, \overline{N(\mu)}) \cong \text{Hom}_{B_{m,1,n}}(\overline{M(\lambda^\sigma)}, \overline{N(\mu^\sigma)})$$

Proof. The condition on the parameter implies that

$$\bar{\delta}_k = \bar{\delta}_{ip+k}$$

for all i and $0 \leq k \leq p-1$. Therefore, by the un-oriented version of [BCD13, Corollary 5.5.2] outlined in the Appendix, we have that rotating both partitions by ip places results in the required isomorphism. \square

Remark 4.0.4. Note that the case that $\overline{M(\lambda)}$ is simple is excluded, as we may only use [BCD13, Corollary 5.5.2] for modules $\overline{M(\lambda)}$ with a Δ -filtration.

Lemma 4.0.5. *Let $M(\lambda^r)$ and $N(\mu^q)$ be simple, standard, or projective modules labelled by $\lambda^r, \mu^q \in \Lambda(m, p, n)$. We have the following isomorphism:*

$$\mathrm{Hom}_{B_{m,p,n}}(M(\lambda^r), N(\mu^q)) \cong \mathrm{Hom}_{B_{m,p,n}}(M(\lambda^{r+1}), N(\mu^{q+1}))$$

Proof. This follows by twisting both modules under conjugation by t_1^d . \square

We let $\epsilon_{r,q}$ denote the Kronecker delta of r and q modulo $\mathrm{hcf}(p/t, p/u)$.

Theorem 4.0.6. *Let $\lambda^r, \mu^q \in \Lambda(m, p, n)$. Let $M(\lambda^r)$ be a standard or projective module labelled by λ^r . Let $N(\mu^q)$ be a simple, standard, or projective module labelled by μ^q . We have isomorphisms*

$$\begin{aligned} \mathrm{Hom}_{B_{m,p,n}}(M(\lambda^r), N(\mu^q)) &\cong \epsilon_{r,q} \mathrm{Hom}_{B_{m,1,n}}(\overline{M(\lambda)}, \oplus_{\rho \in (\mathbb{Z}/p\mathbb{Z})/\langle \sigma^u, \sigma^t \rangle} \overline{N(\mu^\rho)}) \\ &\cong \epsilon_{r,q} \mathrm{Hom}_{B_{m,1,n}}(\oplus_{\rho \in (\mathbb{Z}/p\mathbb{Z})/\langle \sigma^u, \sigma^t \rangle} \overline{M(\lambda^\rho)}, \overline{N(\mu)}). \end{aligned}$$

Proof. We first focus on the righthand side. By Clifford theory, we have that

$$\begin{aligned} \mathrm{Hom}_{B_{m,p,n}}(\oplus_{0 \leq i < p/t} M(\lambda^i), N(\mu^q)) &\cong \mathrm{Hom}_{B_{m,p,n}}(\overline{M(\lambda)} \downarrow, N(\mu^q)) \\ &\cong \mathrm{Hom}_{B_{m,1,n}}(\overline{M(\lambda)}, N(\mu^q) \uparrow) \\ &\cong \mathrm{Hom}_{B_{m,1,n}}(\overline{M(\lambda)}, \oplus_{\rho \in (\mathbb{Z}/p\mathbb{Z})/\langle \sigma^u \rangle} \overline{N(\mu^\rho)}), \\ &\cong \oplus_{\rho \in (\mathbb{Z}/p\mathbb{Z})/\langle \sigma^u \rangle} \mathrm{Hom}_{B_{m,1,n}}(\overline{M(\lambda)}, \overline{N(\mu^\rho)}). \end{aligned}$$

Therefore by Lemma 4.0.3, we have that

$$\mathrm{Hom}_{B_{m,p,n}}(\oplus_{0 \leq i < p/t} M(\lambda^i), N(\mu^q)) \cong \bigoplus_{\substack{\rho \in (\mathbb{Z}/p\mathbb{Z})/ \\ \langle \sigma^u, \sigma^t \rangle}} \mathrm{Hom}_{B_{m,1,n}}(\overline{M(\lambda)}, \overline{N(\mu^\rho)})^{u/\mathrm{hcf}(t,u)}.$$

We now focus on the lefthand side. Any $B_{m,p,n}$ -homomorphism must restrict to a $G(m, p, n)$ -homomorphism, therefore

$$\mathrm{Hom}_{B_{m,p,n}}(\oplus_{0 \leq i < p/t} M(\lambda^i), N(\mu^q)) \cong \mathrm{Hom}_{B_{m,p,n}}(\oplus_{r=q \bmod \mathrm{hcf}(p/t, p/u)} M(\lambda^r), N(\mu^q))$$

as all the other hom-spaces are zero, by Proposition 3.4.2. By repeated application of Lemma 4.0.5, we get that all the summands on the righthand side are isomorphic, and so

$$\mathrm{Hom}_{B_{m,p,n}}(\oplus_{0 \leq i < p/t} M(\lambda^i), N(\mu^q)) \cong \mathrm{Hom}_{B_{m,p,n}}(M(\lambda^r), N(\mu^q))^{u/\mathrm{hcf}(t,u)},$$

therefore the results follows. \square

5. DECOMPOSITION NUMBERS FOR $B_{m,p,n}$

We now use Theorem 4.0.6 and Brauer–Humphrey’s reciprocity to calculate the decomposition numbers for the Brauer algebras of type $G(m, p, n)$. For $m, p, n \in \mathbb{N}$, we let

$$d_{\lambda^r, \mu^q}^{m,p,n}(\delta) = [\Delta_n(\lambda^r) : L_n(\mu^q)]$$

denote the multiplicity of $L_n(\mu^q)$ in $\Delta_n(\lambda^r)$ as a $B_{m,p,n}$ -module. By Brauer–Humphrey’s reciprocity

$$d_{\lambda^r, \mu^q}^{m,p,n}(\delta) = \dim_k(\text{Hom}(P_n(\mu^q), \Delta_n(\lambda^r))).$$

In [Mar, CD11], the decomposition numbers, $d_{\lambda, \mu}^{1,1,n}$, for the classical Brauer algebra (i.e. the type $G(1, 1, n)$ case) are given by the corresponding parabolic Kazhdan–Lusztig polynomials of type (D_n, A_{n-1}) .

The type $G(m, 1, n)$ is covered in [BCD13]. In [BCD13, Appendix] it is shown that the decomposition numbers for the un-oriented cyclotomic Brauer algebras are as follows:

$$d_{\lambda, \mu}^{m,1,n}(\delta) = \prod_{0 \leq i < m} d_{\lambda_i, \mu_i}^{1,1,n}(\bar{\delta}_i).$$

By Theorem 4.0.6, Brauer–Humphrey’s reciprocity, and the above, we have the following description of the decomposition numbers of Brauer algebras of type $G(m, p, n)$.

Theorem 5.0.7. *The decomposition numbers, $d_{\lambda^r, \mu^q}^{m,p,n}(\delta)$ for $B_{m,p,n}$ over a field of characteristic zero are as follows:*

$$d_{\lambda^r, \mu^q}^{m,p,n}(\delta) = \epsilon_{r,q} \sum_{\substack{\rho \in (\mathbb{Z}/p\mathbb{Z})/ \\ \langle \sigma^u, \sigma^t \rangle}} d_{\lambda, \mu^\rho}^{m,1,n}(\delta).$$

Remark 5.0.8. In [BCD13, Remark 5.5.3] it is noted that one can reduce the calculation of certain higher extension groups for $B(m, 1, n)$ to the case of the classical Brauer algebra, as we did above for the decomposition numbers. In these cases one can calculate the corresponding higher extension groups for $B_{m,p,n}$ in a similar fashion to the above.

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REFERENCES

- [AMR06] S. Ariki, A. Mathas, and H. Rui, *Cyclotomic Nazarov–Wenzl algebras*, Nagoya Math. J. **182** (2006), 47–134.
- [BCD13] C. Bowman, A. Cox, and M. De Visscher, *Decomposition numbers for the cyclotomic Brauer algebras in characteristic zero*, J. Algebra **378** (2013), 80–102.

- [BW89] J. Birman and H. Wenzl, *Braids, link polynomials and a new algebra*, Trans. AMS **313** (1989), 249–273.
- [CD11] A. G. Cox and M. De Visscher, *Diagrammatic Kazhdan-Lusztig theory for the (walled) Brauer algebra*, J. Algebra **340** (2011), 151–181.
- [CDDM08] A. G. Cox, M. De Visscher, S. Doty, and P. P. Martin, *On the blocks of the walled Brauer algebra*, J. Algebra **320** (2008), 169–212.
- [CDM09] A. G. Cox, M. De Visscher, and P. P. Martin, *The blocks of the Brauer algebra in characteristic zero*, Representation Theory **13** (2009), 272–308.
- [CFW09] A. Cohen, B. Frenk, and D. Wales, *Brauer algebras of simply laced type*, Israel J. of Mathematics **173** (2009), 335–365.
- [CLY] A. Cohen, S. Liu, and S. Yu, *Brauer algebra of type C*, preprint (2011) available at [arXiv:1101.3416](https://arxiv.org/abs/1101.3416).
- [CMPX06] A. G. Cox, P. P. Martin, A. E. Parker, and C. Xi, *Representation theory of towers of recollement: theory, notes, and examples*, J. Algebra **302** (2006), 340–360.
- [HO01] R. Häring-Oldenburg, *Cyclotomic Birman-Murakami-Wenzl algebras*, J. Pure and Applied Algebra **161** (2001), 113–144.
- [Koi89] K. Koike, *On the decomposition of tensor products of the representations of classical groups: by means of universal characters*, Adv. in Math. **74** (1989), 57–86.
- [Mar] P. P. Martin, *The decomposition matrices of the Brauer algebra over the complex field*, preprint (2009) available at <http://arxiv.org/pdf/0908.1500>.
- [MM10] I. Marin and J. Michel, *Automorphisms of complex reflection groups*, Represent. Theory **14** (2010), 747–788.
- [RR85] I. Reiten and C. Riedtmann, *Skew group algebras in the representation theory of Artin algebras*, J. Algebra (1985), 224–282.
- [Tur89] V. Turaev, *Operator invariants of tangles and R-matrices*, Izvestija AN SSSR ser. math. **53** (1989), 1073–1107, (in Russian).

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